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DENAVIT-HARTENBERG PARAMETERS OF EULER-ANGLE-JOINTS FOR ORDER (N) RECURSIVE FORWARD DYNAMICS

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ABSTRACT

Euler angles are generally used for representing rigid body rotation in three dimensions. In this paper we introduce a concept of Euler-angle-joints (EAJs) which are nothing but three revolute joints so connected by imaginary links with zero length to represent particular Euler angle set. These EAJs can be represented using the well-known Denavit-Hartenberg (DH) parameters. The proposed EAJs are useful in representing a spherical joint present in any multibody system. One can then derive a corresponding decoupled natural orthogonal complement (DeNOC) matrices used in dynamic formulation to obtain the analytical expressions of the generalized inertia matrix elements in scalar form. These expressions are used to develop an O(n) — n being the number of degree-of- freedom of a serial chain — recursive forward dynamics algorithm. The methodology suggested is illustrated with a numerical example.

1 INTRODUCTION

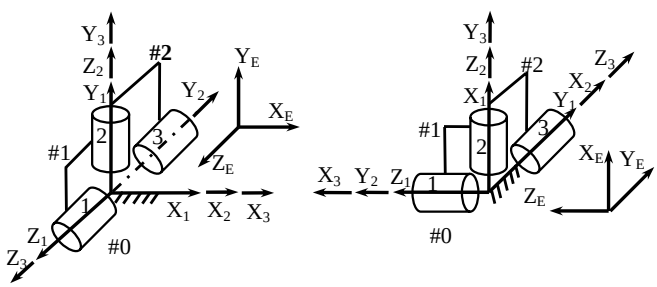
Euler angles [1] represent the orientation of rigid body in space. It has wide acceptability in the field of robotics, aerospace, bio-mechanics, etc., as three independent parameters are used to represent a rotation in three dimensions. On the other hand the use of one degree-of-freedom (DOF) joints, revolute and prismatic, and their axis representation using well-known Denavit-Hartenberg (DH) [2] parameters to define a link configuration, is wide spread in robotics applications. Euler angle rotations cause by an equivalent set of three intersecting revolute joint are termed here as, Euler-angle-joints (EAJ). Note that, so far no formal way of representing Euler angles rotations using DH parameters is reported in literature. Hence it is formally introduced and their benefits are explored. It is then used to represent spherical joint, which helped to obtain recursive O(n) forward dynamics algorithm based on the decoupled natural orthogonal complement (DeNOC) matrices [3]. Typically, DeNOC matrices are used to eliminate the reaction forces and moments in the un-coupled Newton-Euler equation of motions. They tend to provide benefits like analytical recursive expressions for the elements of the vectors and matrices appearing in the dynamic equations of motion, recursive O(n) forward dynamics algorithm, etc.

There are mainly two types of simulation algorithms, namely, O(n^3) and O(n). An O(n^3) forward dynamics algorithm finds the inverse of the generalized inertia matrix (GIM) using some decomposition algorithm, say, Cholesky or Gaussian elimination [4]. A recursive O(n) algorithm, on the other hand, is inversion method wherein expressions of the inverted GIM are found analytically. Several such algorithms [3, 5-12] were proposed in the literature for serial manipulators with revolute and prismatic joints. For example, Featherstone [5] introduced a concept of articulated body inertia which leads to an O(n) algorithm. Saha [6] also proposed an algorithm for a serial multibody systems based on reverse Gaussian elimination of the GIM. An O(n) algorithm is reported to be efficient [7] than order O(n^3) for n>7, and also numerically stable [8] due to smooth acceleration plots. The main objective of this paper is to introduce the concept of Euler-angle-joints (EAJ) and their representation using the DH parameters to replace a spherical joint in a multibody system to obtain an O(n) recursive forward dynamic algorithm.

2 DH PARAMETERS FOR EULER-ANGLE-JOINTS

Euler angles are preferred over other rotation representations, namely direction cosines or Euler parameter due to their minimal set [13]. Even though Euler angle suffers from numerical singularity, there are ways to overcome the same [14, 15]. In this paper we introduce three revolute joints so connected by imaginary link with zero length to represent a particular set of Euler angles, i.e., ZYZ, ZXZ, etc. These revolute joints are referred here as Euler-angle- joints (EAJs). The EAJs architecture can then be described using well-known

DH parameters. It can be shown that for every Euler angle set EAJs can be formed and corresponding DH parameters be obtained. For example, Fig. 1 shows two such representations, namely, symmetric Euler angle set ZYZ and asymmetric set ZXY. For ZYZ set, first revolute joint is parallel to Z_E connects an imaginary link #1 to the fixed base #0. Joint 2 parallel to Y_E connects another imaginary link #2 with #1. Finally joint 3 parallel to Z_E connects a real link with link #2. Now, the co-ordinate frames are assigned on the imaginary links based on the DH notations defined in [16]. Frame 1 is the fixed frame and successive frames are denoted as 2, 3, etc. while frame E is attached to the moving rigid body.



(a) ZYZ Euler angle set (b) ZXY Euler angle set

Figure 1: DH representation of Euler angles

The DH Parameters for the ZYZ and ZXY EAJs for the zero configurations are given in Table 1.

Table 1: DH parameter for ZYZ and ZXY EAJs

(a) ZYZ EAJs

	θ_i	α_i	a_i	b_i
1	$\theta_1(0)$	-90	*0*	0
2	$\theta_2(0)$	90	*0*	0
3	$\theta_3(0)$	0	*0*	0

(b) ZXY EAJs

	θ_i	α_i	a_i	b_i
1	$\theta_1(90)$	90	*0*	0
2	$\theta_2(90)$	90	*0*	0
3	$\theta_3(90)$	90	*0*	0

θ_i is angle between X_i and X_{i+1} about Z_i ; a_i is link length
α_i is angle between Z_i and Z_{i+1} about X_{i+1}; b_i is joint off-set

Note that the rotation matrix using DH parameters is given by [16]

$$\mathbf{Q}_i \equiv \begin{bmatrix} C\theta_i & -S\theta_i C\alpha_i & S\theta_i S\alpha_i \\ S\theta_i & C\theta_i C\alpha_i & -C\theta_i S\alpha_i \\ 0 & S\alpha_i & C\alpha_i \end{bmatrix} \quad \text{where } i = 1, 2, 3 \quad (1)$$

The overall rotation matrix between frame 1 and E for ZYZ Euler angle set derived next. Putting the value of θ_i and α_i, for i = 1, 2, 3, from Table 1(a) in (1), one obtains subsequent elementary rotation matrices as

$$\mathbf{Q}_1 \equiv \begin{bmatrix} C\theta_1 & 0 & -S\theta_i \\ S\theta_1 & 0 & C\theta_i \\ 0 & -1 & 0 \end{bmatrix}, \ \mathbf{Q}_2 \equiv \begin{bmatrix} C\theta_2 & 0 & S\theta_2 \\ S\theta_2 & 0 & -C\theta_2 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\text{and } \mathbf{Q}_3 \equiv \begin{bmatrix} C\theta_3 & -S\theta_3 & 0 \\ S\theta_3 & C\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (2)$$

Now overall rotation matrix from frame 1 to frame E, can then be obtained as $\mathbf{Q} \equiv \mathbf{Q}_1\ \mathbf{Q}_2\ \mathbf{Q}_3$ whose expression is given bellow:

$$\mathbf{Q} \equiv \begin{bmatrix} C\theta_1 C\theta_2 C\theta_3 - S\theta_1 S\theta_3 & -C\theta_1 C\theta_2 S\theta_3 - S\theta_1 C\theta_3 & C\theta_1 S\theta_2 \\ S\theta_1 C\theta_2 C\theta_3 + C\theta_1 S\theta_3 & -S\theta_1 C\theta_2 S\theta_3 + C\theta_1 C\theta_3 & S\theta_1 S\theta_2 \\ -S\theta_2 C\theta_3 & S\theta_2 S\theta_3 & C\theta_2 \end{bmatrix} \quad (3)$$

The rotation matrix in (3) is exactly same as the one obtained using ZYZ Euler angle [16]. Similarly it can be shown for asymmetric Euler angle set ZXY and others. Here EAJs are further utilized to represent a spherical joint in a multibody system which provides benefits like analytical expressions for the elements of the associated matrices in scalar level and O(n) recursive dynamic algorithm.

3 DYNAMIC FORMULATION

In this section, expressions for kinematic constraints due to joints, including the spherical one, are derived which lead to the definition of the DeNOC matrices for a serial multibody system. Furthermore, the matrices are utilized to develop the equations of motion and analytical expression of associated generalized inertia matrix (GIM).

3.1 Spherical joint representation

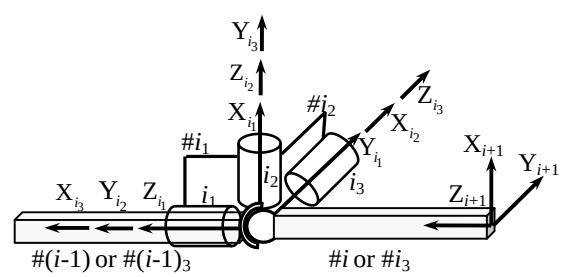


Figure 2: A spherical joint representation

Here the proposed Euler-angle-joints (EAJ) are used to represent a spherical joint using ZXY set as shown in Fig. 2. Two consecutive links #(i-1) and #i are considered to be connected by three orthogonally placed revolute joints i_1, i_2 and i_3 connecting two imaginary links (#i_1 and #i_2) of the zero lengths and masses. The #i_3^{th} link is the actual physical link #i which is attached to the third revolute joint, i.e., i_3. Note that as such there is no special reason of choosing ZXY set except that

unlike ZYZ in Fig. 1(a), it is not singularity as shown in Fig. 1(b) in zero configuration. This avoids the trouble of initiating the computation algorithm Moreover, no discussion on the singularity of EAJs will be made in this paper as most of the physical joints have restricted motion and one suitably chose the EAJ set for not hitting singularity. In worst scenario one may use the singularity avoidance algorithms reported in literature [14, 15].

3.2 Kinematic Constraint

Kinematic constraint equations due to the EAJs, Fig. 2, are written in terms of velocity of the origin of the links and its angular velocity. Let us consider $\boldsymbol{\omega}_{i_j}$ and $\dot{\mathbf{o}}_{i_j}$ are the 3-dimensional vectors of the angular velocity and linear velocity of the origin of the #i_j link. Now the constraint equations for the ith spherical joint that connects #(i-1) and #i links, i.e., i_1, i_2 and i_3 revolute joints connecting #i_1 and #i_2 imaginary links are written as

$$\text{Link \#}i_1\text{:}\ \ \boldsymbol{\omega}_{i_1} = \boldsymbol{\omega}_{(i-1)_3} + \dot{\theta}_{i_1}\mathbf{e}_{i_1}\ ;\ \dot{\mathbf{o}}_{i_1} = \dot{\mathbf{o}}_{i-1} + \boldsymbol{\omega}_{(i-1)_3}\times\mathbf{a}_{i-1} \tag{4a}$$

$$\text{Link \#}i_j\text{:}\ \ \boldsymbol{\omega}_{i_j} = \boldsymbol{\omega}_{i_{j-1}} + \dot{\theta}_{i_j}\mathbf{e}_{i_j}\ ;\ \dot{\mathbf{o}}_{i_j} = \dot{\mathbf{o}}_{i_{j-1}},\ j=2,3 \tag{4b}$$

Next, we define the 6-dimensional vector of twist associated with the angular and linear velocities of link #i_j as, $\mathbf{t}_{i_j} \equiv [\,\boldsymbol{\omega}_{i_j}^T \quad \dot{\mathbf{o}}_{i_j}^T\,]^{\mathrm{T}}$. Accordingly, the twists of the successive bodies are obtained as

$$\text{Link \#}i_1\text{:}\ \ \mathbf{t}_{i_1} = \mathbf{A}_{i,i-1}\mathbf{t}_{(i-1)_3} + \mathbf{p}_{i_1}\dot{\theta}_{i_1} \tag{5a}$$

$$\text{Link \#}i_j\text{:}\ \ \mathbf{t}_{i_j} = \mathbf{t}_{i_{j-1}} + \mathbf{p}_{i_j}\dot{\theta}_{i_j},\ j=2,3 \tag{5b}$$

In (5a), $\mathbf{A}_{i,i\text{-}1}$ is the 6×6 twist propagation matrix, and $\mathbf{p}_{i_j}$ is the 6-dimensional motion propagation vector [3], which are given by

$$\mathbf{A}_{i,i\text{-}1} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \tilde{\mathbf{a}}_{i,i-1} & \mathbf{1} \end{bmatrix},\ \mathbf{p}_{i_j} \equiv \begin{bmatrix} \mathbf{e}_{i_j} \\ \mathbf{0} \end{bmatrix} \tag{6}$$

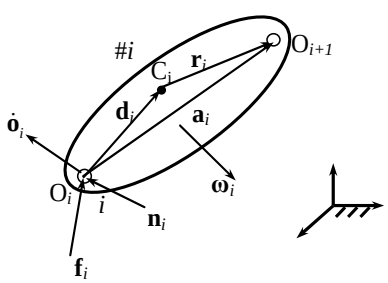


Figure 3: Free body diagram of link #i

In (6), $\mathbf{O}$ and $\mathbf{1}$ denote the 3×3 zero and identity matrices respectively. Also $\tilde{\mathbf{a}}_{i,i\text{-}1} = -\tilde{\mathbf{a}}_{i\text{-}1}$ is the skew symmetric matrix associated with the vector $\mathbf{a}_{i\text{-}1}$. Vector $\mathbf{a}_{i\text{-}1}$ is similar to vector $\mathbf{a}_i$ of the link #i as indicated in Fig. 3. Moreover, vector $\mathbf{e}_{i_j}$ is the unit vector parallel to the axis of rotation of the i_j revolute joint. Note in (5b) that there is no twist propagation matrix as it is an identity. Let us now introduce the definition of the generalized twist, $\mathbf{t} \equiv [\,\mathbf{t}_{1_1}^{\mathrm{T}} \quad \mathbf{t}_{1_2}^{\mathrm{T}} \ldots \quad \mathbf{t}_{n_3}^{\mathrm{T}}\,]^{\mathrm{T}}$ and generalized joint rate $\dot{\boldsymbol{\theta}} \equiv [\,\dot{\theta}_{i_1} \quad \dot{\theta}_{i_2} \quad \ldots \quad \dot{\theta}_{n_3}\,]^{\mathrm{T}}$ for n coupled links with spherical joints only. Then using (5a-b) the expression for the generalized twist, $\mathbf{t}$, is obtained as

$$\mathbf{t} = \mathbf{N}_l\mathbf{N}_d\dot{\boldsymbol{\theta}} \tag{7}$$

Matrices $\mathbf{N}_l$ and $\mathbf{N}_d$ are the desired DeNOC matrices for the spatial multibody chain consisting of spherical joints only. The 18n×18n lower block triangular matrix $\mathbf{N}_l$ and 18n×3n block diagonal matrix $\mathbf{N}_d$ have the following representation:

$$\mathbf{N}_l = \begin{bmatrix} \bar{\mathbf{1}}_{11} & \mathbf{O} & \cdots & \mathbf{O} \\ \bar{\mathbf{A}}_{21} & \bar{\mathbf{1}}_{22} & \cdots & \mathbf{O} \\ \vdots & \ddots & \ddots & \vdots \\ \bar{\mathbf{A}}_{n1} & \cdots & \bar{\mathbf{A}}_{n,n\text{-}1} & \bar{\mathbf{1}}_{nn} \end{bmatrix} \text{and } \mathbf{N}_d = \begin{bmatrix} \mathbf{P}_1 & \mathbf{O} & \cdots & \mathbf{O} \\ \mathbf{O} & \mathbf{P}_2 & \cdots & \mathbf{O} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{O} & \mathbf{O} & \cdots & \mathbf{P}_n \end{bmatrix} \tag{8}$$

In above expression $\bar{\mathbf{A}}_{i,i-1}$, $\bar{\mathbf{1}}_{ii}$, and $\mathbf{P}_i$ are the 18×18, 18×18 and 18×3 matrices written as

$$\bar{\mathbf{A}}_{i,i-1} \equiv \begin{bmatrix} \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} \\ \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} \\ \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} \end{bmatrix},\ \bar{\mathbf{1}}_{ii} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} & \mathbf{O} \\ \mathbf{1} & \mathbf{1} & \mathbf{O} \\ \mathbf{1} & \mathbf{1} & \mathbf{1} \end{bmatrix} \text{ and}$$

$$\mathbf{P}_i \equiv diag\begin{bmatrix} \mathbf{p}_{i_1} & \mathbf{p}_{i_2} & \mathbf{p}_{i_3} \end{bmatrix} \tag{9}$$

The block partitioned form of $\mathbf{N}_l$ and $\mathbf{N}_d$ is unique if only origin is selected as reference point to define the twist of a link. Note that in the presence of revolute or universal joint between two real links the expression of $\mathbf{N}_l$ and $\mathbf{N}_d$ remains same but sizes of its content block matrices, $\bar{\mathbf{A}}_{i,i-1}$, $\bar{\mathbf{1}}_{ii}$ and $\mathbf{P}_i$ reduce. For example if i^{th} joint is universal instead of spherical, i.e., EAJs, then, 12×18 $\bar{\mathbf{A}}_{i,i-1}$, 12×12 $\bar{\mathbf{1}}_{ii}$ and 12×2 $\mathbf{P}_i$ are as follows:

$$\bar{\mathbf{A}}_{i,i-1} \equiv \begin{bmatrix} \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} \\ \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} \end{bmatrix},\ \bar{\mathbf{1}}_{ii} \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{1} & \mathbf{1} \end{bmatrix}$$

$$\text{and } \mathbf{P}_i \equiv diag\begin{bmatrix} \mathbf{p}_{i_1} & \mathbf{p}_{i_2} \end{bmatrix} \tag{10a}$$

Similarly if i^{th} joint is revolute where as others are EAJs then, 6×18 $\bar{\mathbf{A}}_{i,i-1}$, 6×6 $\bar{\mathbf{1}}_{ii}$ and 6×1 $\mathbf{P}_i$ are

$$\bar{\mathbf{A}}_{i,i-1} \equiv \begin{bmatrix} \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} & \mathbf{A}_{i,i-1} \end{bmatrix},\ \mathbf{P}_i \equiv \mathbf{p}_{i_1},\ \text{and } \bar{\mathbf{1}}_{ii} \equiv \mathbf{1} \tag{10b}$$

In case two universal or two revolute joints appear one after another in the kinematic chain, then corresponding changes in the sizes of $\bar{\mathbf{A}}_{i,i-1}$ will occur.

3.3 Dynamics Using the DeNOC Matrices

The Newton-Euler equations of motion [3] of rigid link #*i* can be written as

$$\mathbf{M}_i\dot{\mathbf{t}}_i + \mathbf{\Omega}_i\mathbf{M}_i\mathbf{E}_i\mathbf{t}_i = \mathbf{w}_i\,,\ \text{where } \mathbf{w}_i = \mathbf{w}_i^E + \mathbf{w}_i^C + \mathbf{w}_i^G \qquad (11)$$

where $\mathbf{\Omega}_i$, $\mathbf{M}_i$ and $\mathbf{E}_i$ are the 6×6 angular velocity, extended mass and coupling matrices for the ith link, which are defined as

$$\mathbf{\Omega}_i \equiv \begin{bmatrix} \tilde{\boldsymbol{\omega}}_i & \mathbf{O} \\ \mathbf{O} & \tilde{\boldsymbol{\omega}}_i \end{bmatrix},\ \mathbf{M}_i \equiv \begin{bmatrix} \mathbf{I}_i & m_i\tilde{\mathbf{d}}_i \\ -m_i\tilde{\mathbf{d}}_i & m_i\mathbf{1} \end{bmatrix} \text{ and } \mathbf{E}_i \equiv \begin{bmatrix} \mathbf{1} & \mathbf{O} \\ \mathbf{O} & \mathbf{O} \end{bmatrix} \qquad (12)$$

In (12) $\tilde{\boldsymbol{\omega}}_i$ and $\tilde{\mathbf{d}}_i$ are the skew symmetric matrices associated with the 3-dimensional vectors, $\boldsymbol{\omega}_i$ and $\mathbf{d}_i$, respectively, whereas, $\mathbf{I}_i$ and m_i are the 3×3 inertia tensor about O_i, and mass of the ith body, respectively. Note that the 6-dimensional wrench $\mathbf{w}_i$, is defined as $\mathbf{w}_i \equiv [\,\mathbf{n}_i^T \quad \mathbf{f}_i^T\,]^{\mathrm{T}}$ — $\mathbf{n}_i$ and $\mathbf{f}_i$ being the moment and force applied on the ith link, respectively as shown in Fig. 3. Moreover $\mathbf{w}_i^E$, $\mathbf{w}_i^C$, and $\mathbf{w}_i^G$ are the wrenches due to external moment and force, constraints, and gravity respectively. Now for 3*n* coupled bodies comprising of both real and imaginary links, the equations of motion can be expressed as

$$\mathbf{M}\dot{\mathbf{t}} + \mathbf{\Omega}\mathbf{M}\mathbf{E}\mathbf{t} = \mathbf{w}^E + \mathbf{w}^C + \mathbf{w}^G \qquad (13)$$

In above equation, matrices $\mathbf{\Omega}$, $\mathbf{M}$ and $\mathbf{E}$ are the 18*n*×18*n* generalized matrices of angular velocity, extended mass and coupling matrix, respectively, which are defined as

$$\mathbf{M} \equiv \mathrm{diag}\left[\mathbf{M}_{1_1} \quad \dots \quad \mathbf{M}_{n_3}\right], \mathbf{\Omega} \equiv \mathrm{diag}\,[\mathbf{\Omega}_{1_1} \quad \cdots \quad \mathbf{\Omega}_{n_3}]$$
$$\text{and } \mathbf{E} \equiv \mathrm{diag}\left[\mathbf{E}_{1_1} \quad \cdots \quad \mathbf{E}_{n_3}\right] \qquad (14)$$

Pre-multiplying (13) by the transpose of the DeNOC matrices a minimal set of equations of motion eliminating the constraint wrenches is obtained, i.e.,

$$\mathbf{N}_d^T\mathbf{N}_l^T\left(\mathbf{M}\dot{\mathbf{t}} + \mathbf{\Omega}\mathbf{M}\mathbf{E}\mathbf{t}\right) = \mathbf{N}_d^T\mathbf{N}_l^T(\mathbf{w}^E + \mathbf{w}^G) \qquad (15)$$

where $\mathbf{N}_d^T\mathbf{N}_l^T\mathbf{w}^C$ vanishes [3]. Substituting (7) and its derivatives into (15) and rearranging the terms, one can obtain following:

$$\mathbf{I}\ddot{\boldsymbol{\theta}} + \mathbf{C}\dot{\boldsymbol{\theta}} = \boldsymbol{\tau} + \boldsymbol{\tau}^G \qquad (16)$$

where the expressions of the generalized inertia matrix (GIM) **I**, matrix of convective inertia (MCI) **C**, generalized external force $\boldsymbol{\tau}$, and the generalized force due to gravity $\boldsymbol{\tau}^G$ are given as

$$\mathbf{I} = \mathbf{N}^T\mathbf{M}\mathbf{N}\,,\ \mathbf{C} = \mathbf{N}^T(\mathbf{M}\dot{\mathbf{N}} + \mathbf{\Omega}\mathbf{M}\mathbf{E}\mathbf{N})\,,\ \boldsymbol{\tau} = \mathbf{N}^T\mathbf{w}^E\,,\ \boldsymbol{\tau}^G = \mathbf{N}^T\mathbf{w}^G \qquad (17)$$

where $\mathbf{N} \equiv \mathbf{N}_l\mathbf{N}_d$ is used

3.4 Analytical Expressions of the generalized Inertia matrix

The O(*n*) recursive algorithm presented in this paper is based on the analytical expression of the GIM. The GIM of (17) is written as

$$\mathbf{I} = \mathbf{N}_d^T\tilde{\mathbf{M}}\mathbf{N}_d \ \text{ where } \tilde{\mathbf{M}} = \mathbf{N}_l^T\mathbf{M}\mathbf{N}_l \qquad (18)$$

In (18) the matrix $\tilde{\mathbf{M}}$ is referred as the generalized composite mass matrix. Substituting $\mathbf{N}_l$ and **M** from (8) and (12) into (18), analytical expression of the block elements of $\tilde{\mathbf{M}}$ is obtained for *i* = 1…*n*, *j* = 1…*i* and *k*, *l* =1…3 as

$$\tilde{\mathbf{M}}(i_k, j_l) = \tilde{\mathbf{M}}_{i_k}\mathbf{A}_{i,j}\,,\ \textit{if}\ \ i = j;\ \mathbf{A}_{i,j} = \mathbf{1} \qquad (19)$$

where $\tilde{\mathbf{M}}_{i_k}$ is the 6×6 matrix of composite body connected by bodies i_j….*n*. It can be obtained recursively for *i* = *n*…1 as

$$\text{Link \#}i_3\text{:} \qquad \tilde{\mathbf{M}}_{i_3} = \mathbf{M}_{i_3} + \mathbf{A}_{i+1,i}^T\tilde{\mathbf{M}}_{(i+1)_1}\mathbf{A}_{i+1,i} \qquad (20a)$$

$$\text{Link \#}i_j\text{:} \qquad \tilde{\mathbf{M}}_{i_j} = \tilde{\mathbf{M}}_{i_{j+1}}\,,\ j = 2, 1 \qquad (20b)$$

Next the analytical expressions for the scalar elements of the GIM using (18) are obtained for *i* = 1…*n*, *j* = 1…*i* and *k*, *l* = 1…3 as,

$$I(i_k, j_l) = \mathbf{p}_{i_k}^T\tilde{\mathbf{M}}_{i_k}\mathbf{A}_{i,j}\mathbf{p}_{j_l}\,,\ \textit{if}\ \ i = j;\ \mathbf{A}_{i,j} = \mathbf{1} \qquad (21)$$

In case the original definition of Euler angles are used, the analytical expressions for elements of the GIM would be 3×3 matrix as opposed to scalar quantity obtained in (20). This is due to the fact that the motion propagation similar to **p** of (6) would be matrix of 6×3. The advantage of having scalar term in (20) is that the GIM can now be decomposed analytically to obtain full O(*n*) recursive forward dynamics algorithm [3,6].

4 RECURSIVE FORWARD DYNAMICS ALGORITHM

In this section, recursive steps which lead to recursive O(*n*) forward dynamics for the serial multibody systems with spherical joint are presented. The algorithm is based on the $\mathbf{UDU}^T$ decomposition of the GIM proposed in [6]. The analytical expression for each element of **U** and **D** are available using (21) after decomposition. Now the equation of motion of (16) can be rewritten as

$$\mathbf{UDU}^T\ddot{\boldsymbol{\theta}} = \boldsymbol{\varphi}\text{, where } \mathbf{I} = \mathbf{UDU}^T \text{ and } \boldsymbol{\varphi} = \boldsymbol{\tau} + \boldsymbol{\tau}^G - \mathbf{C}\dot{\boldsymbol{\theta}} \quad (22a)$$

In (22a) $\boldsymbol{\varphi}$ is obtained recursively using any inverse dynamics algorithm, say, [3]. The joint accelerations are then solved as

$$\ddot{\boldsymbol{\theta}} = \mathbf{U}^{-T}\mathbf{D}^{-1}\mathbf{U}^{-1}\boldsymbol{\varphi} \quad (22b)$$

Next, the three steps and the associated expressions to obtain the joint accelerations from the dynamic equations of the motion are given below:

Step 1: Find $\hat{\boldsymbol{\varphi}} = \mathbf{U}^{-1}\boldsymbol{\varphi}$ (for $i = n.....1$)

Link #i_3: $$\hat{\varphi}_{i_3} = \varphi_{i_3} - \mathbf{p}_{i_3}^T\bar{\boldsymbol{\eta}}_{i_3} \quad (23a)$$

where the 6-dimensional vector, $\bar{\boldsymbol{\eta}}_{i_3}$ is obtained recursively as

$$\bar{\boldsymbol{\eta}}_{i_3} = \mathbf{A}_{i+1,i}^T\boldsymbol{\eta}_{(i+1)_1} \text{ and } \boldsymbol{\eta}_{(i+1)_1} = \boldsymbol{\psi}_{(i+1)_1}\hat{\varphi}_{(i+1)_1} + \bar{\boldsymbol{\eta}}_{(i+1)_1} \quad (23b)$$

Link #i_j: (j = 2, 1) $$\hat{\varphi}_{i_j} = \varphi_{i_j} - \mathbf{p}_{i_j}^T\bar{\boldsymbol{\eta}}_{i_j} \quad (24a)$$

where the 6-dimensional vector, $\bar{\boldsymbol{\eta}}_{i_j}$ is obtained recursively as

$$\bar{\boldsymbol{\eta}}_{i_j} = \boldsymbol{\eta}_{i_{j+1}} \text{ and } \boldsymbol{\eta}_{i_{j+1}} = \boldsymbol{\psi}_{i_{j+1}}\hat{\varphi}_{i_{j+1}} + \bar{\boldsymbol{\eta}}_{i_{j+1}} \quad (24b)$$

In (23b) and (24b), $\boldsymbol{\eta}_{(i+1)_1}$ and $\boldsymbol{\eta}_{i_{j+1}}$ are the 6-dimensional vectors and $\mathbf{A}_{i+1,i}^T$ is represented similarly as in (6)

Step 2: Find $\tilde{\boldsymbol{\varphi}} = \mathbf{D}^{-1}\hat{\boldsymbol{\varphi}}$ (for $i = 1.....n$)

Link #i_j (j = 1, 2, 3) $$\tilde{\varphi}_{i_j} = \hat{\varphi}_{i_j} / \hat{m}_{i_j} \quad (25)$$

Step 3: Find $\ddot{\boldsymbol{\theta}} = \mathbf{U}^{-T}\tilde{\boldsymbol{\varphi}}$, (for $i = 1.....n$)

Link #i_1: $$\ddot{\theta}_{i_1} = \tilde{\varphi}_{i_1} - \boldsymbol{\psi}_{i_1}^T\bar{\boldsymbol{\mu}}_{i_1} \quad (26a)$$

where the 6-dimensional vector $\bar{\boldsymbol{\mu}}_{i_1}$ is obtained recursively as

$$\bar{\boldsymbol{\mu}}_{i_1} = \mathbf{A}_{i,i-1}\boldsymbol{\mu}_{(i-1)_3}\text{, where } \boldsymbol{\mu}_{(i-1)_3} = \mathbf{p}_{(i-1)_3}\ddot{\theta}_{(i-1)_3} + \bar{\boldsymbol{\mu}}_{(i-1)_3} \quad (26b)$$

Link #i_j :(j = 2, 3) $$\ddot{\theta}_{i_j} = \tilde{\varphi}_{i_j} - \boldsymbol{\psi}_{i_j}^T\bar{\boldsymbol{\mu}}_{i_j} \quad (27a)$$

where the 6-dimensional vector $\bar{\boldsymbol{\mu}}_{i_j}$ is obtained recursively as

$$\bar{\boldsymbol{\mu}}_{i_j} = \boldsymbol{\mu}_{i_{j-1}} \text{ where } \boldsymbol{\mu}_{i_{j-1}} = \mathbf{p}_{i_{j-1}}\ddot{\theta}_{i_{j-1}} + \bar{\boldsymbol{\mu}}_{i_{j-1}} \quad (27b)$$

In (26b) and (27b), $\boldsymbol{\mu}_{(i-1)_3}$ and $\boldsymbol{\mu}_{i_{j-1}}$ are the 6-dimensional vector and $\mathbf{A}_{i,i-1}$ is represented similarly as in (6). Also note that the 6-dimensional vector $\boldsymbol{\psi}_{i_j}$ of (23b), (24b), (26a) and (27a), and $\hat{m}_{i_j}$ of (25) are obtained as

$$\boldsymbol{\psi}_{i_j} = \frac{\hat{\boldsymbol{\psi}}_{i_j}}{\hat{m}_{i_j}}\text{, where } \hat{m}_{i_j} = \mathbf{p}_{i_j}^T\hat{\boldsymbol{\psi}}_{i_j} \text{ and } \hat{\boldsymbol{\psi}}_{i_j} = \hat{\mathbf{M}}_{i_j}\mathbf{p}_{i_j} \quad (28)$$

In (28) the 6×6 matrix $\hat{\mathbf{M}}_{i_j}$ represents the mass and inertia properties of the articulated body i_j, and obtained recursively for $i = n.....1$ as

$$\hat{\mathbf{M}}_{i_j} = \mathbf{M}_{i_j} + \mathbf{A}_{i+1,i}^T\hat{\mathbf{M}}_{(i+1)_1,(i+1)_1}\mathbf{A}_{i+1,i}\text{, if } j=3 \quad (29a)$$

$$\hat{\mathbf{M}}_{i_j} = \hat{\mathbf{M}}_{i_{j+1},i_{j+1}}\text{, if } j = 1, 2 \quad (29b)$$

where

$$\hat{\mathbf{M}}_{i_j,i_j} = \hat{\mathbf{M}}_{i_j} - \hat{\boldsymbol{\psi}}_{i_j}\boldsymbol{\psi}_{i_j}^T\text{, } j = 1, 2, 3 \quad (30)$$

Note that if we use original definition of Euler angles instead of EAJs, **p** has matrix form of 6×3. Hence, $\hat{m}$ is not scalar in (28). Instead it will be a 3×3 matrix. As a result, $\boldsymbol{\psi}$ would require O(3^3) steps which would have been computationally more disadvantageous.

5 A NUMERICAL EXAMPLE

A numerical illustration is presented here as an academic interest to predict the spatial behavior of a double pendulum with spherical joints. The double pendulum using ZXY EAJs is shown in Fig. 4. The DH frames are assigned similar to Fig. 2 and DH parameters as given in Table 1(b).

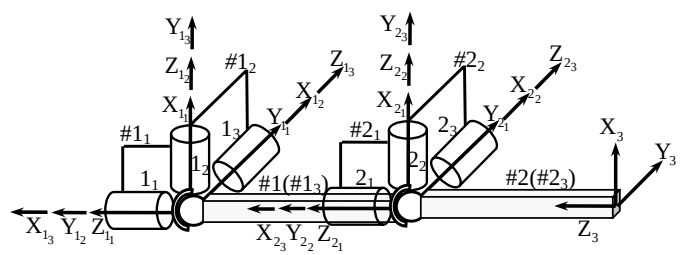


Figure 4: A Double pendulum using EAJs

The model parameter, i.e. length and mass, for double pendulum are taken as $l_1 = l_2 = 0.5$ m, and $m_1 = m_2 = 10$ Kg. The initial joint angles and joint rates are given in Table 2.The joint angle are kept such that double pendulum has spatial behavior. Joint accelerations are first found using the recursive O(n) forward dynamics presented in the previous section. They are then integrated using '*ode45*' function of Matlab [17]. Corresponding results for the joint angles and rates are shown in Figs. 5-6. The angle $\theta_{i,i}$ in Fig. 5 corresponds to rotation about joint i_j, similarly in Fig. 6 the joint rate $\dot{\theta}_{i,j}$ corresponds to the joint rate for joint i_j .The comparison of the results with

those obtained using ADAMS [18] software is also shown in Figs. 5-6 using mark ‘o’. The close match of the results validates the concept of EAJs proposed in this paper for the dynamics modeling and simulation of multibody system with spherical joins.

Table 2: Initial joint angles and joint rates

	1_1	1_2	1_3	2_1	2_2	2_3
θ_i	90	90	30	90	60	90
$\dot{\theta}_i$	0	0	0	0	0	0

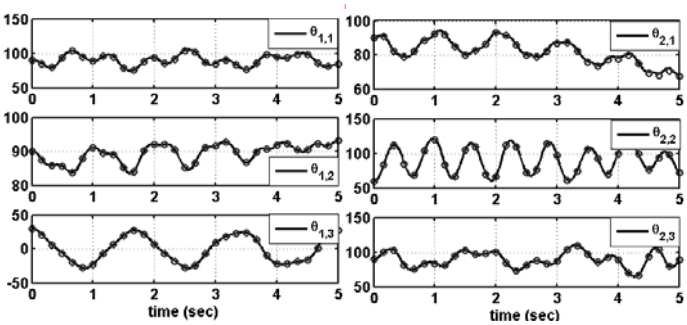


Figure 5: Angles (degree) at joint 1 and 2

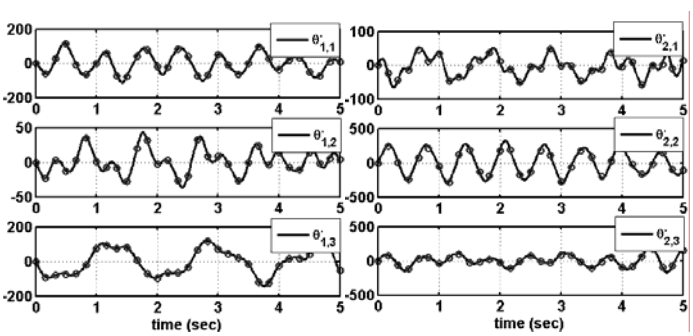


Figure 6: Rates (degree/sec) at joint 1 and 2

6 CONCLUSIONS

A concept of Euler-angle-joints (EAJs) along with its DH parameters is introduced in this paper to represent a spherical joint. Even though three intersecting revolute joints are used in robotics literature to represent a spherical joint but no formal correlation with the Euler angle set is ever made. Hence proposed EAJs are considered a contribution of this paper. The unique form of the DeNOC matrices for dynamics is derived and further employed to find the analytical expression for the elements of the generalized inertia matrix (GIM) arising out of constrained equations of motion of a serial multibody system with spherical joints. The derivation helped to generate recursive O(n) — n being the number of degree-of-freedom of a serial chain — dynamics algorithm in contrast to the representation using the original definition of Euler angles as explained after (30). Finally the representation proposed here leads to a unified representation of one-, two- and three-degree-of-freedom joints.

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